

Modules Satisfying the Prime Radical Condition and a Sheaf Construction for Modules $I^{*\dagger\ddagger}$

M. Behboodi^{a,b§} and M. Sabzevari^a

^aDepartment of Mathematical Sciences, Isfahan University of Technology
P.O.Box: 84156-83111, Isfahan, Iran

^bSchool of Mathematics, Institute for Research in Fundamental Sciences (IPM)
P.O.Box: 19395-5746, Tehran, Iran
mbehbood@cc.iut.ac.ir
sabzevari@math.iut.ac.ir

Abstract

The purpose of this paper and its sequel, is to introduce a new class of modules over a commutative ring R , called \mathbb{P} -radical modules (modules M satisfying the prime radical condition “ $(\sqrt[\mathbb{P}]{\mathcal{P}M} : M) = \mathcal{P}$ ” for every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$, where $\sqrt[\mathbb{P}]{\mathcal{P}M}$ is the intersection of all prime submodules of M containing $\mathcal{P}M$). This class contains the family of primeful modules properly. This yields that over any ring all free modules and all finitely generated modules lie in the class of \mathbb{P} -radical modules. Also, we show that if R is a domain (or a Noetherian ring), then all projective modules are \mathbb{P} -radical. In particular, if R is an Artinian ring, then all R -modules are \mathbb{P} -radical and the converse is also true when R is a Noetherian ring. Also an R -module M is called \mathbb{M} -radical if $(\sqrt[\mathbb{M}]{\mathcal{M}M} : M) = \mathcal{M}$; for every maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$. We show that the two concepts \mathbb{P} -radical and \mathbb{M} -radical are equivalent for all R -modules if and only if R is a Hilbert ring. Semisimple \mathbb{P} -radical (\mathbb{M} -radical) modules are also characterized. In Part II we shall continue the study of this construction, and as an application, we show that the sheaf theory of spectrum of \mathbb{P} -radical modules (with the Zariski topology) resembles to that of rings.

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[§]Corresponding author.

1 Introduction

All rings in this article are commutative with identity and modules are unital. For a ring R we denote by $\dim(R)$ the classical Krull dimension of R and for a submodule N of an R -module M we denote the annihilator of the factor module M/N by $(N : M)$, i.e., $(N : M) = \{r \in R \mid rM \subseteq N\}$. We call M faithful if $(0 : M) = 0$.

The theory sheaf of rings (with the Zariski topology) on the spectrum of prime ideals of a commutative ring is one of the main tools in Algebraic Geometry. Recall that the *spectrum* $\text{Spec}(R)$ of a ring R consists of all prime ideals of R and is non-empty. For each ideal I of R , we set $V(I) = \{\mathcal{P} \in \text{Spec}(R) : I \subseteq \mathcal{P}\}$. Then the sets $V(I)$, where I is an ideal of R , satisfy the axioms for the closed sets of a topology on $\text{Spec}(R)$, called the *Zariski topology* of R . The distinguished open sets of $\text{Spec}(R)$ are the open sets of the form $D(f) = \{\mathcal{P} \in \text{Spec}(R) : f \notin \mathcal{P}\} = \text{Spec}(R) \setminus V(f)$, where $V(f) = V(Rf)$. These sets form a basis for the Zariski topology on $\text{Spec}(R)$ (see for examples, Atiyah and Macdonald [1] and Hartshorne [14]).

We recall that a *sheaf of rings* \mathcal{O}_X on a topological space X is an assignment of a ring $\mathcal{O}_X(U)$ to each open set U in X , together with, for each inclusion $U \subseteq V$ a *restriction homomorphism* $\text{res}_{V,U} : \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$, subject to the following conditions:

- (i) $\mathcal{O}_X(\emptyset) = 0$.
- (ii) $\text{res}_{U,U} = \text{id}_U$.
- (iii) If $U \subseteq V \subseteq W$, then $\text{res}_{V,U} \circ \text{res}_{W,V} = \text{res}_{W,U}$.
- (iv) For each open cover $\{U_\alpha\}$ of $U \subseteq X$ and for each collection of elements $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that for all α, β if $\text{res}_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \text{res}_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$, then there is a unique $f \in \mathcal{O}_X(U)$ such that for all α , $f_\alpha = \text{res}_{U, U_\alpha}(f)$.

We think of the elements of $\mathcal{O}_X(U)$ as “functions” defined on U . The restriction homomorphisms correspond to restricting a function on a big open set to a smaller one.

It is well-known that for any commutative ring R , there is a sheaf of rings on $\text{Spec}(R)$, called the *structure sheaf*, denoted by $\mathcal{O}_{\text{Spec}(R)}$, defined as follows: for each prime ideal \mathcal{P} of R , let $R_{\mathcal{P}}$ be the localization of R at \mathcal{P} . For an open set $U \subseteq \text{Spec}(R)$, we define $\mathcal{O}_{\text{Spec}(R)}(U)$ to be the set of functions $s : U \rightarrow \coprod_{\mathcal{P} \in U} R_{\mathcal{P}}$, such that $s(\mathcal{P}) \in R_{\mathcal{P}}$, for each $\mathcal{P} \in U$, and such that s is locally a quotient of elements of R : to be precise, we require that for each $\mathcal{P} \in U$, there is a neighborhood V of \mathcal{P} , contained in U , and elements $a, f \in R$, such that for each $\mathcal{Q} \in V$, $f \notin \mathcal{Q}$, and $s(\mathcal{Q}) = \frac{a}{f}$ in $R_{\mathcal{Q}}$ (see for example Hartshorne [14], for definition and basic properties of the sheaf $\mathcal{O}_{\text{Spec}(R)}$).

Let R be a ring. Then an R -module $M \neq 0$ is called a *prime module* if $rm = 0$ for $m \in M$, $r \in R$ implies that $m = 0$ or $rM = (0)$ (i.e., $r \in \text{Ann}(M)$). We call a proper

submodule P of an R -module M to be a *prime submodule* of M if M/P is a prime module, i.e., whenever $rm \in P$, then either $m \in P$ or $rM \subseteq P$ for every $r \in R$, $m \in M$. Thus P is a prime submodule (or \mathcal{P} -prime submodule) of M if and only if $\mathcal{P} = \text{Ann}(M/P)$ is a prime ideal of R and M/P is a torsion free R/\mathcal{P} -module. This notion of prime submodule was first introduced and systematically studied in [8, 11] and recently has received a good deal of attention from several authors; see for examples [3, 5, 6, 15, 19, 20, 21, 23]. The set of all prime submodules of M is called the *Spectrum* of M and denoted by $\text{Spec}(M)$. For any submodule N of M , we have a set $V(N) = \{P \in \text{Spec}(M) \mid (N : M) \subseteq (P : M)\}$. Then the sets $V(N)$, where N is a submodule of M satisfy the axioms for the closed sets of a topology on $\text{Spec}(M)$, called the Zariski topology of M (see for examples [16, 17, 22, 25]).

For an R -module M let $\psi : \text{Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$ defined by $\psi(P) = (P : M)/\text{Ann}(M)$ for every $P \in \text{Spec}(M)$. ψ is called the natural map of $\text{Spec}(M)$. An R -module M is called *primeful* if either $M = (0)$ or $M \neq (0)$ and the natural map $\psi : \text{Spec}(M) \rightarrow \text{Spec}(R/\text{Ann}(M))$ is surjective. This notion of primeful module has been extensively studied by Lu [18]).

Let M be a primeful faithful R -module. In [25], the author obtained an R -module $\mathcal{O}_X(U)$ for each open set U in $X = \text{Spec}(M)$ such that \mathcal{O}_X is a sheaf of modules over X . In fact, \mathcal{O}_X is a generalization of the structure sheaf of rings to primeful faithful modules.

The purpose of this paper and its sequel, is to develop the structure sheaf of rings to a wider class of modules called \mathbb{P} -radical modules ((modules M satisfying the prime radical condition “ $(\sqrt[\mathbb{P}]{\mathcal{P}M} : M) = \mathcal{P}$ ” for every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$, where $\sqrt[\mathbb{P}]{\mathcal{P}M}$ is the intersection of all prime submodules of M containing $\mathcal{P}M$ but, if M has no prime submodule, then $\sqrt[\mathbb{P}]{\mathcal{P}M} = M$). In Section 1, we introduce and study \mathbb{P} -radical modules. Several characterizations of \mathbb{P} -radical are given in Proposition 2.1. It is shown that the class of \mathbb{P} -radical modules contains the class of primeful R -modules properly (Proposition 2.3 and Example 2.4). This yields over any ring all free modules, all finitely generated modules and all homogeneous semisimple modules lie in the class of \mathbb{P} -radical modules. Moreover, if R is a Noetherian ring (or an integral domain), then every projective R -modules is \mathbb{P} -radical (see Theorem 2.5 and Corollary 2.4(iii)). In particular, if R is an Artinian ring, then all R -modules are \mathbb{P} -radical (Theorem 2.13). Although the converse is not true in general (Example 2.14), but we show that the converse is true for certain classes of rings like Noetherian rings and integral domains (see Theorem 2.16 and Corollary 2.17). Also, an R -module M is called \mathbb{M} -radical module if $(\sqrt[\mathbb{M}]{\mathcal{M}M} : M) = \mathcal{M}$; for every maximal ideal \mathcal{M} containing $\text{Ann}(M)$. Thus, we have the following chart of implications for M :

$$M \text{ is finitely generated} \Rightarrow M \text{ is primeful} \Rightarrow M \text{ is } \mathbb{P}\text{-radical} \Rightarrow M \text{ is } \mathbb{M}\text{-radical}$$

We show that the two concepts \mathbb{P} -radical and \mathbb{M} -radical are equivalent for all R -modules if

and only if R is a Hilbert ring (Theorem 2.11). Also the two concepts primful and \mathbb{M} -radical are equivalent for all R -modules if and only if $\dim(R) = 0$ (Theorem 2.12). Recall that an R -module M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$ (see for example [2, 10] for more details). Proposition 2.18, suggests that for a multiplication module M the four concepts “finitely generated”, “primeful”, “ \mathbb{P} -radical” and “ \mathbb{M} -radical” are equivalent. Also we prove the Proposition 2.19 that is analogue of Nakayamas Lemma for \mathbb{P} -radical modules. In Section 3 semisimple primful modules, semisimple \mathbb{P} -Radical modules and semisimple \mathbb{M} -Radical modules, are fully investigated. For instance, it is shown that a semisimple R -module M is a \mathbb{M} -radical if and only if there exists a submodule N of M such that $N \cong \bigoplus_{\text{Ann}(M) \subseteq \mathcal{M} \in \text{Max}(R)} R/\mathcal{M}$ (Proposition 3.5.) Also, a semisimple R -module M is a \mathbb{P} -radical if and only if M is a \mathbb{M} -radical module and $R/\text{Ann}(M)$ is a Hilbert ring (Proposition 3.6). In Part II we shall continue the study of this construction and, as an application, we show that the sheaf theory of spectrum of \mathbb{P} -radical modules (with Zariski topology) resembles to that of rings.

2 \mathbb{P} -Radical and \mathbb{M} -Radical Modules

Unlike the rings with identity, not every R -module contains a prime submodule; for example \mathbb{Z}_p^∞ as \mathbb{Z} -module does not contain a prime submodule (see [5] or [11]). Let N be a proper submodule of an R -module M . Then the *prime radical* $\sqrt[{}]{N}$ is the intersection of all prime submodules of M containing N or, in case there are no such prime submodules, $\sqrt[{}]{N}$ is M . Clearly $V(N) = V(\sqrt[{}]{N})$. We note that, for each ideal I of R , $\sqrt[{}]{I} = \sqrt{I}$ (the intersection of all prime ideals of R containing I). The prime radical of submodules are studied by several authors; see for example [4, 5, 19].

The problem asking which R -module M satisfies the equality $\sqrt[{}]{IM} = \sqrt{I}M$ for every ideal I containing $\text{Ann}(M)$, had been investigated for finitely generated modules in [19]. Also, in [18], the author, extend the investigation to primeful flat content modules (e.g. free modules), and primeful flat modules over rings with Noetherian spectrum.

In this article we introduce a slightly different of the above equality; that is the equality $(\sqrt[{}]{IM} : M) = \sqrt{I}$ for every ideal I containing $\text{Ann}(M)$. This prime radical condition of modules play a key role in our investigation for give a sheaf construction for modules.

The following proposition offer several characterizations of R -modules M satisfies the equality $(\sqrt[{}]{IM} : M) = \sqrt{I}$ for every ideal I containing $\text{Ann}(M)$.

Proposition 2.1 *For an R -module M the following statements are equivalent:*

- (1) $(\sqrt[p]{IM} : M) = \sqrt{I}$ for each ideal $I \supseteq \text{Ann}(M)$.
- (2) $(\sqrt[p]{PM} : M) = \mathcal{P}$ for each prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$.
- (3) $\sqrt{I} = \bigcap_{P \in V(IM)} (P : M)$, for every ideal $I \supseteq \text{Ann}(M)$.
- (4) $\mathcal{P} = \bigcap_{P \in V(PM)} (P : M)$, for every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$.

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (1). Let $I \supseteq \text{Ann}(M)$. Clearly, $I \subseteq (IM : M) \subseteq (\sqrt[p]{IM} : M)$ and $\sqrt[p]{IM} : M$ is a radical ideal (i.e., is an intersection of all prime ideals). Thus $\sqrt{I} \subseteq (\sqrt[p]{IM} : M)$. On the other hand,

$$(\sqrt[p]{IM} : M) \subseteq \bigcap_{\mathcal{P} \in V(I)} (\sqrt[p]{PM} : M) = \bigcap_{\mathcal{P} \in V(I)} \mathcal{P} = \sqrt{I}.$$

Thus, $(\sqrt[p]{IM} : M) = \sqrt{I}$.

(1) \Rightarrow (3). Let $I \supseteq \text{Ann}(M)$. It is easy to check that

$$\sqrt{I} = (\sqrt[p]{IM} : M) = ((\bigcap_{P \in V(IM)} P) : M) = \bigcap_{P \in V(IM)} (P : M).$$

(3) \Rightarrow (4) is trivial.

(4) \Rightarrow (2). Suppose that \mathcal{P} is a prime ideal of R such that $\mathcal{P} \supseteq \text{Ann}(M)$. Since $\bigcap_{P \in V(PM)} P = \sqrt[p]{PM}$ and $\bigcap_{P \in V(PM)} (P : M) = (\bigcap_{P \in V(PM)} P : M)$, we conclude that $\mathcal{P} = (\sqrt[p]{PM} : M)$. \square

Definition 2.2 . Let M be an R -module M . Then M is called a \mathbb{P} -radical module if M satisfies the equivalent conditions listed in the above Proposition 2.1.

The following proposition shows that for any ring R , all primeful modules are \mathbb{P} -radical.

Proposition 2.3 Let R be a ring. Then any primeful R -module M is a \mathbb{P} -radical module.

Proof. Let $\mathcal{P} \supseteq \text{Ann}(M)$. Since M is a primeful module, there exists $P \in \text{Spec}(M)$ such that $(P : M) = \mathcal{P}$. It follows that $\sqrt[p]{PM} \subseteq P$ and $\mathcal{P} = \sqrt{\mathcal{P}} \subseteq (\sqrt[p]{PM} : M)$. Thus

$$\mathcal{P} = \sqrt{\mathcal{P}} \subseteq (\sqrt[p]{PM} : M) \subseteq (P : M) = \mathcal{P},$$

and hence $(\sqrt[p]{PM} : M) = \mathcal{P}$. Thus M is a \mathbb{P} -radical module. \square

Let R be a ring. It is easy to see that every free R -module M is primeful. Also in [16, Theorem 2.2], it is shown that every finitely generated R -module M is primeful. In particular, if R is a domain, then every projective R -module is primeful (see [16, Corollary 4.3]). Thus by using these facts and above proposition, we have the following corollary.

Corollary 2.4 *Let R be a ring.*

- (i) *Every free R -module is \mathbb{P} -radical.*
- (ii) *Every finitely generated R -module is \mathbb{P} -radical.*
- (iii) *If R is a domain, then every projective R -module is \mathbb{P} -radical.*

Next, we show that Corollary 2.4 (iii) is also true when we replace “ R is a domain” with “ R is a Noetherian ring”.

Theorem 2.5 *Let R be a Noetherian ring. Then every projective R -module is \mathbb{P} -radical.*

Proof. Suppose that M is a projective R -module and $\mathcal{P} \supseteq \text{Ann}(M)$. We claim that $\mathcal{P}M \neq M$. If $\mathcal{P}M = M$, then we put $\mathcal{A} = \{I \supseteq R \mid IM \neq M \text{ and } \text{Ann}(M) \subseteq I \subseteq \mathcal{P}\}$. Then $\text{Ann}(M) \in \mathcal{A}$ and so $\mathcal{A} \neq \emptyset$. Since R is Noetherian, \mathcal{A} has a maximal element, say \mathcal{P}_0 . Then \mathcal{P}_0 is a prime ideal of R , otherwise, there exist $a, b \in R \setminus \mathcal{P}_0$ such that $ab \in \mathcal{P}_0$. It follows that $(\mathcal{P}_0 + Ra)M = (\mathcal{P}_0 + Rb)M = M$ and so $M = (\mathcal{P}_0 + Ra)(\mathcal{P}_0 + Rb)M \subseteq \mathcal{P}_0M$, a contradiction. It is well-known that for each projective R -module M and each ideal I of R , the factor module M/IM is also projective as an R/I -module. Thus $\bar{M} := M/\mathcal{P}_0M$ is projective as an $\bar{R} := R/\mathcal{P}_0$ -module. Since \bar{R} is a domain, by Corollary 2.4 (iii), \bar{M} is a \mathbb{P} -radical \bar{R} -module. If $r \in R \setminus \mathcal{P}_0$, then $(Rr + \mathcal{P}_0)M = M$ and it follows that $r + \mathcal{P}_0 \notin \text{Ann}_{\bar{R}}(\bar{M})$, i.e., $\text{Ann}_{\bar{R}}(\bar{M}) = (0)$. Since $\bar{\mathcal{P}} := \mathcal{P}/\mathcal{P}_0$ is a prime ideal of \bar{R} we must have $(\sqrt[\mathbb{P}]{\bar{\mathcal{P}}\bar{M}} : \bar{M}) = \bar{\mathcal{P}}$. But, the equality $\mathcal{P}M = M$ implies that $\bar{\mathcal{P}}\bar{M} = \bar{M}$ and so $(\sqrt[\mathbb{P}]{\bar{\mathcal{P}}\bar{M}} : \bar{M}) = \bar{R}$, a contradiction. Thus $\mathcal{P}M \neq M$ and so $\mathcal{P}M$ is a proper submodule of M . Suppose $F = M \oplus L$ where F is a free R -module and L is a submodule of F . Clearly $\mathcal{P}F$ is a prime submodule of F , i.e., $F/\mathcal{P}F$ is a prime R -module and $\text{Ann}(F/\mathcal{P}F) = \mathcal{P}$. Since $F/\mathcal{P}F \cong M/\mathcal{P}M \oplus L/\mathcal{P}L$, we conclude that $M/\mathcal{P}M$ is also a prime R -module with $\text{Ann}(M/\mathcal{P}M) = \text{Ann}(F/\mathcal{P}F) = \mathcal{P}$, i.e., $\mathcal{P}M$ is a prime submodule of M with $(\mathcal{P}M : M) = \mathcal{P}$. Thus $(\sqrt[\mathbb{P}]{\mathcal{P}M} : M) = (\mathcal{P}M : M) = \mathcal{P}$. Therefore M is a \mathbb{P} -radical R -module. \square

The following example shows that the converse of Proposition 2.3 is not true in general (see also next Proposition 2.7). Thus, the class of \mathbb{P} -radical modules contains the class of primeful R -modules properly.

Example 2.6 (See also [18], P. 136, Example 1.) Let $M = \bigoplus_{p \in \Omega} \mathbb{Z}/p\mathbb{Z}$ as a \mathbb{Z} -module where Ω is the set of prime integers. Clearly $\text{Ann}(M) = 0$ and according to [18], for any non-zero prime ideal (p) of \mathbb{Z} , pM is a (p) -prime submodule of M , but it doesn't have any (0) -prime submodule. Thus M is not a primeful \mathbb{Z} -module. Clearly the zero submodule (0) of M is an intersection of maximal submodules of M and hence $\sqrt[\mathbb{P}]{(0)} = (0)$. Thus

$(\sqrt[p]{(0)M} : M) = ((0) : M) = \text{Ann}(M) = (0)$. Also, if (q) is a nonzero prime (maximal) ideal of \mathbb{Z} , then $(q)M = \bigoplus_{q \neq p \in \Omega} \mathbb{Z}/p\mathbb{Z} \neq M$. It follows that $(q) = (\sqrt[p]{(q)M} : M)$. Thus M is a \mathbb{P} -radical module and it is not a primeful module.

Proposition 2.7 *Let R be a ring. Then there exists an R -module M which is a \mathbb{P} -radical module but it is not a primeful module if and only if there exist prime ideals \mathcal{P} and $\{\mathcal{P}_i\}_{i \in I}$ of R such that $\mathcal{P} \subsetneq \mathcal{P}_i$ (for each $i \in I$) and $\mathcal{P} = \bigcap_{i \in I} \mathcal{P}_i$.*

Proof. (\Rightarrow) Let M be an R -module which is a \mathbb{P} -radical module but it is not a primeful module. Assume that \mathcal{P} is a prime ideal of R such that M has not any \mathcal{P} -prime submodule. Thus for any prime submodule N of M with $\mathcal{P}M \subseteq N$, we must have $\mathcal{P} \subsetneq \mathcal{P}_N := (N : M)$. Since M is a \mathbb{P} -radical module,

$$\mathcal{P} = (\sqrt[p]{\mathcal{P}M} : M) = \left(\bigcap_{\mathcal{P}M \subseteq N \in \text{Spec}(M)} N : M \right) = \bigcap_{\mathcal{P}M \subseteq N \in \text{Spec}(M)} (N : M) = \bigcap_{\mathcal{P}M \subseteq N \in \text{Spec}(M)} \mathcal{P}_N.$$

(\Leftarrow) Without loss of generality we can assume that $\mathcal{P} = \bigcap_{i \in I} \mathcal{P}_i$ and for each prime ideal $\mathcal{Q} \supsetneq \mathcal{P}$ of R there exists $i \in I$ such that $\mathcal{Q} = \mathcal{P}_i$. Let $M = \bigoplus_{i \in I} R/\mathcal{P}_i$ as R -module. Clearly $\text{Ann}(M) = \bigcap_{i \in I} \mathcal{P}_i = \mathcal{P}$ and for each $j \in I$, $N_j = \bigoplus_{j \neq i \in I} R/\mathcal{P}_i$ is a \mathcal{P}_j -prime submodule of M with $\mathcal{P}M \subseteq \mathcal{P}_jM \subseteq N_j$. Thus for each prime ideal $\mathcal{P}_j \supsetneq \text{Ann}(M) = \mathcal{P}$, $(\sqrt[p]{\mathcal{P}_jM} : M) = \mathcal{P}_j$ (since $\mathcal{P}_j \subseteq (\sqrt[p]{\mathcal{P}_jM} : M) \subseteq (N_j : M) = \mathcal{P}_j$). On the other hand

$$(\sqrt[p]{\mathcal{P}M} : M) \subseteq \left(\bigcap_{i \in I} N_i : M \right) = \bigcap_{i \in I} (N_i : M) = \bigcap_{i \in I} \mathcal{P}_i = \mathcal{P}.$$

Thus M is a \mathbb{P} -radical module. We claim that M doesn't have any \mathcal{P} -prime submodule, for if not, let N be a \mathcal{P} -prime submodule of M with $(N : M) = \mathcal{P}$. Since $N \neq M$, there exists $j \in I$ such that $(\dots, 0, 1 + \mathcal{P}_j, 0, \dots) \notin N$. Since $\mathcal{P}_j(\dots, 0, 1 + \mathcal{P}_j, 0, \dots) \in N$, we must have $\mathcal{P}_jM \subseteq N$, i.e., $\mathcal{P}_j \subseteq \mathcal{P}$, a contradiction. Thus M is not a primeful R -module. \square

By Proposition 2.1 and Definition 2.2, an R -module M is a \mathbb{P} -radical module if and only if $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$ for each prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$. Now we have to adapt the notion of \mathbb{M} -radical modules which generalized \mathbb{P} -radical modules.

Definition 2.8 . Let M be an R -module M . Then M is called an \mathbb{M} -radical module (or a *Maxful module*) if $(\sqrt[p]{\mathcal{M}M} : M) = \mathcal{M}$ for each maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$.

The following evident proposition offers several other characterizations of \mathbb{M} -radical modules.

Proposition 2.9 *Let R be a ring and M a nonzero R -module. Then the following statements are equivalent:*

- (1) M is an \mathbb{M} -radical module.
- (2) $\mathcal{M}M \neq M$ for every maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$.
- (3) $\mathcal{P}M \neq M$ for every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$.
- (4) For every maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$ there exists a maximal submodule P of M such that $(P : M) = \mathcal{M}$.
- (5) For every maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$ there exists a prime submodule P of M such that $(P : M) = \mathcal{M}$.

Proof. (1) \Rightarrow (2). Clearly for each maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$, the equality $(\sqrt[p]{\mathcal{M}M} : M) = \mathcal{M}$ implies that $\mathcal{M}M \neq M$.

(2) \Rightarrow (1). Suppose $\mathcal{M}M \neq M$ for every maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$. Then for each maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$, $\mathcal{M}M$ is a prime submodule of M and so $\sqrt[p]{\mathcal{M}M} = \mathcal{M}M$. It follows that $\mathcal{M} \subseteq (\sqrt[p]{\mathcal{M}M} : M) = (\mathcal{M}M : M) = \mathcal{M}$, i.e., $(\sqrt[p]{\mathcal{M}M} : M) = \mathcal{M}$. Thus M is an \mathbb{M} -radical module.

(2) \Leftrightarrow (3) is clear.

(2) \Rightarrow (5). Suppose that $\mathcal{M} \supseteq \text{Ann}(M)$ is a maximal ideal. Then $\mathcal{M}M \neq M$ implies that $\mathcal{M}M$ is a prime submodule with $(\mathcal{M}M : M) = \mathcal{M}$.

(5) \Rightarrow (2) is clear.

(4) \Rightarrow (5) is clear (since every maximal submodule is a prime submodule).

(5) \Rightarrow (4). Suppose that $(P : M) = \mathcal{M}$ where P is a prime submodule of M and $\mathcal{M} \supseteq \text{Ann}(M)$ is a maximal ideal. Then M/P is an R/\mathcal{M} -vector space and hence M/P has a maximal R/\mathcal{M} -subspace say K/P . Clearly $K \subseteq M$ is a maximal R -submodule and $(K : M) = \mathcal{M}$. \square

A commutative ring R is called a *Hilbert ring*, also *Jacobson* or *Jacobson-Hilbert* ring, if every prime ideal of R is the intersection of maximal ideals. The class of commutative Hilbert rings is closed under forming finite polynomial rings. For an integral extension $R \subseteq S$, the ring S is Hilbert if and only if R is a Hilbert ring. The main interest in Hilbert rings in commutative algebra and algebraic geometry is their relation with Hilbert's Nullstellensatz (see Goldman [12], Cortzen and Small [7], Theorem 1, and [9], Theorem 4.19, for more details).

Next, we will show that the two concepts \mathbb{P} -radical and \mathbb{M} -radical are equivalent for all R -modules if and only if R is a Hilbert ring. As we have observed earlier some modules M have no prime submodules (for example \mathbb{Z}_{p^∞} as \mathbb{Z} -module) and we call them *primeless*. We recall that a module M over a domain R is called *divisible* if $rM = M$ for

each $0 \neq r \in R$ and M is called *torsion* if $\text{Ann}(m) \neq 0$ for each $m \in M$.

We need the following evident lemma.

Lemma 2.10 *Let R be a domain. Then any torsion divisible R -module is primeless.*

Theorem 2.11 *Let R be a ring. Then the two concepts \mathbb{P} -radical and \mathbb{M} -radical are equivalent for all R -modules if and only if R is a Hilbert ring.*

Proof. (\Rightarrow) . Let every \mathbb{M} -radical module is a \mathbb{P} -radical module. To obtain a contradiction, suppose that \mathcal{P} is a prime ideal of R such that it is not an intersection of maximal ideals, i.e., $\mathcal{P} \neq \bigcap_{\mathcal{M} \in \text{Max}(R)} \mathcal{M}$. It is easy to check that every \mathbb{M} -radical R/\mathcal{P} -module is also a \mathbb{P} -radical R/\mathcal{P} -module. Let $S = R/\mathcal{P}$ and Q be the field of fraction of S . Since \mathcal{P} is not a maximal ideal, S is not a field and so $S \neq Q$. Suppose K is a nonzero proper S -submodule of Q . Then $L = Q/K$ is a torsion divisible S -module and so by Lemma 2.10, L is a primeless S -module. We put

$$M = \left(\bigoplus_{\mathcal{M} \in \text{Max}(S)} S/\mathcal{M} \right) \oplus L$$

as an S -module. Clearly M is a \mathbb{M} -radical S -module, since for each $\mathcal{M}_1 \in \text{Max}(S)$,

$$\mathcal{M}_1 M = \left(\bigoplus_{\mathcal{M}_1 \neq \mathcal{M} \in \text{Max}(S)} S/\mathcal{M} \right) \oplus L$$

is a prime S -module with $(\mathcal{M}_1 M : M) = \mathcal{M}_1$. On the other hand we claim that every prime S -submodule of M is of the above form. To see this, let P be a prime S -submodule of M . We claim that $\bigoplus_{\mathcal{M} \in \text{Max}(S)} S/\mathcal{M} \not\subseteq P$, otherwise we must have $M/P \cong L/T$ for some proper submodule T of L , a contradiction (since L is a primeless S -module). Thus there exists $\mathcal{M}_1 \in \text{Max}(S)$ such that $(0, \dots, 0, 1 + \mathcal{M}_1, 0, \dots) \notin P$. Since $\mathcal{M}_1(0, \dots, 0, 1 + \mathcal{M}_1, 0, \dots) \subseteq P$, $\mathcal{M}_1 M \subseteq P$, and hence $\mathcal{M}_1 M = P$. It follows that

$$\sqrt[p]{(0)} = \bigcap_{\mathcal{M} \in \text{Max}(R)} \mathcal{M} M = L.$$

Since $\text{Ann}_S(L) = (0)$, we conclude that $\text{Ann}_S(M) = (0)$. Thus

$$(\sqrt[p]{(0)}M : M) = (\sqrt[p]{(0)} : M) = (L : M) = ((0) : \bigoplus_{\mathcal{M} \in \text{Max}(S)} S/\mathcal{M}) = \bigcap_{\mathcal{M} \in \text{Max}(S)} \mathcal{M}.$$

But $\bigcap_{\mathcal{M} \in \text{Max}(R)} \mathcal{M} \neq (0)$ since $\mathcal{P} \neq \bigcap_{\mathcal{P} \subseteq \mathcal{M} \in \text{Max}(R)} \mathcal{M}$. Thus $(\sqrt[p]{(0)}M : M) \neq (0)$, which is impossible (since M is a \mathbb{P} -radical module). Thus R is a Hilbert ring.

(\Leftarrow). Since every \mathbb{P} -radical module is a \mathbb{M} -radical module, it suffices to show that every \mathbb{M} -radical module is a \mathbb{P} -radical module. Assume that M is an \mathbb{M} -radical module and \mathcal{P} is a prime ideal of R with $\mathcal{P} \supseteq \text{Ann}(M)$. Since R is a Hilbert ring, $\mathcal{P} = \bigcap_{\mathcal{P} \subseteq \mathcal{M} \in \text{Max}(R)} \mathcal{M}$. On the other hand, since M is an \mathbb{M} -radical module, $(\mathcal{M}M : M) = \mathcal{M}$ for each $\mathcal{P} \subseteq \mathcal{M} \in \text{Max}(R)$. It follows that

$$(\sqrt[p]{\mathcal{P}M} : M) \subseteq \left(\bigcap_{\mathcal{P} \subseteq \mathcal{M} \in \text{Max}(R)} \mathcal{M}M : M \right) = \bigcap_{\mathcal{P} \subseteq \mathcal{M} \in \text{Max}(R)} (\mathcal{M}M : M) = \bigcap_{\mathcal{P} \subseteq \mathcal{M} \in \text{Max}(R)} \mathcal{M} = \mathcal{P}.$$

Since $\mathcal{P} \subseteq (\mathcal{P}M : M) \subseteq (\sqrt[p]{\mathcal{P}M} : M)$, we conclude that $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$. Thus M is a \mathbb{P} -radical module. \square

Let M be an R -module. By Proposition 2.3 and Proposition 2.9, we have the following chart of implications for M :

$$M \text{ is a primeful module} \Rightarrow M \text{ is a } \mathbb{P}\text{-radical module} \Rightarrow M \text{ is a } \mathbb{M}\text{-radical module}$$

In general, none of the implications is reversible. However, for zero dimensional rings, the implications can be replaced by equivalences, as we shall now show.

Theorem 2.12 *Let R be a ring. Then the two concepts primful and \mathbb{M} -radical are equivalent for all R -modules if and only if $\dim(R) = 0$.*

Proof. Assume that the two concepts primful and \mathbb{M} -radical are equivalent for all R -modules. Then by Theorem 2.11, R is a Hilbert ring. Suppose, contrary to our claim, that $\dim(R) \geq 2$ and \mathcal{P} is a non-maximal prime ideal of R . Thus $\mathcal{P} = \bigcap_{\mathcal{P} \subseteq \mathcal{M} \in \text{Max}(R)} \mathcal{M}$. Therefore, by Proposition 2.7, there exists an R -module M which is a \mathbb{P} -radical (\mathbb{M} -radical) module but it is not a primeful module, a contradiction. Thus $\dim(R) = 0$. The converse is clear. \square

The following result shows over an Artinian ring, every module is \mathbb{P} -radical (consequently, every module is primeful and also every module is \mathbb{M} -radical).

Theorem 2.13 *Let R be an Artinian ring. Then every R -module is a \mathbb{P} -radical module.*

Proof. Since $\dim(R) = 0$, by Theorem 2.12, the three concepts primful, \mathbb{P} -radical and \mathbb{M} -radical are equivalent for all R -modules. Since R is Artinian, $R = R_1 \times \cdots \times R_n$, where $n \in \mathbb{N}$ and each R_i is an Artinian local ring. First give the proof for the case $n = 1$, i.e., R is a local ring with maximal ideal \mathcal{M} . Suppose M is a nonzero R -module. Then $\text{Ann}(M) \neq R$ and so $\text{Ann}(M) \subseteq \mathcal{M}$. By Proposition 2.9, it suffices to show that

$\mathcal{M}M \neq M$. If R is a domain (field), then $\mathcal{M} = (0)$ and so the proof is complete. Hence, to obtain a contradiction, suppose that R is not a domain and also $\mathcal{M}M = M$. Then there are nonzero elements $a, b \in R$ such that $ab = 0$. Thus $(Ra)(Rb)M = (0)$ and so either $RaM \neq M$ or $RbM \neq M$. It follows that $\mathcal{A} := \{I \supseteq R \mid IM \neq M \text{ and } I \neq (0)\}$ is a nonempty set of ideals of R . Since R is Noetherian, \mathcal{A} has a maximal element, say \mathcal{P} . Since $\mathcal{P}M \neq M$, \mathcal{P} is not a prime (maximal) ideal of R (i.e., $\mathcal{P} \neq \mathcal{M}$). Thus there exist ideals A, B of R such that $\mathcal{P} \subsetneq A$, $\mathcal{P} \subsetneq B$ and $AB \subseteq \mathcal{P}$. Thus the equalities $AM = BM = M$ implies that $M = ABM \subseteq \mathcal{P}M$, a contradiction. Therefore, in the case R is a local ring every R -module is a \mathbb{P} -radical module. Now assume $n \geq 2$ and for each i ($1 \leq i \leq n$) let \mathcal{M}_i be the maximal ideal of the local ring R_i . Suppose M is a nonzero R -module and $\text{Ann}(M) \subseteq \mathcal{M}$ where \mathcal{M} is a maximal ideal of R . Clearly \mathcal{M} is the form $R_1 \times \cdots \times R_{i-1} \times \mathcal{M}_i \times R_{i+1} \cdots \times R_n$ for some i . Without loss of generality we can assume that $i = 1$, i.e., $\mathcal{M} = \mathcal{M}_1 \times R_2 \times \cdots \times R_n$. Again, by Proposition 2.9, it suffices to show that $\mathcal{M}M = (\mathcal{M}_1 \times R_2 \times \cdots \times R_n)M \neq M$. On the contrary, suppose that $(\mathcal{M}_1 \times R_2 \times \cdots \times R_n)M = M$. Let $I = R_1 \times (0) \times \cdots \times (0)$ and $J = (0) \times R_2 \times \cdots \times R_n$. Then I, J are ideals of R with $J = \text{Ann}(I)$. Thus $R_1 \cong R/J$ and so $\bar{M} = IM$ is an unitary R_1 -module (in fact, $\bar{M} = (R_1 \times 0 \times \cdots \times 0)M$ is an unitary R_1 -module with $r_1\bar{m}$ defined to be $r_1(1, 0, \dots, 0)m$ for $r_1 \in R_1$ and $\bar{m} \in \bar{M}$). We claim that $\bar{M} \neq (0)$, otherwise, $R_1 \times (0) \times \cdots \times (0) \subseteq \text{Ann}(M) \subseteq \mathcal{M}_1 \times R_2 \times \cdots \times R_n$, a contradiction. Thus \bar{M} is a nonzero R_1 -module and so by using case $n = 1$, we have $\mathcal{M}_1\bar{M} \neq \bar{M}$, i.e., $(\mathcal{M}_1 \times 0 \times \cdots \times 0)M \neq (R_1 \times 0 \times \cdots \times 0)M$. On the other hand, for each $m \in M$, $(1, 0, \dots, 0)m \in M = (\mathcal{M}_1 \times R_2 \times \cdots \times R_n)M$. Thus for each $m \in M$,

$$(1, 0, \dots, 0)m = \sum_{j=1}^k (p_{1j}, r_{2j}, \dots, r_{nj})m_j$$

for some $k \in \mathbb{N}$, $m_j \in M$, $p_{1j} \in \mathcal{M}_1$, $r_{ij} \in R_i$ were $2 \leq i \leq n$ and $1 \leq j \leq k$. Thus

$$(1, 0, \dots, 0)m = (1, 0, \dots, 0)^2m = \sum_{j=1}^k (p_{1j}, 0, \dots, 0)m_j \in (\mathcal{M}_1 \times (0) \times \cdots \times (0))M,$$

for each $m \in M$. It follows that $(R_1 \times 0 \times \cdots \times 0)M \subseteq (\mathcal{M}_1 \times (0) \times \cdots \times (0))M$, i.e., $\mathcal{M}_1\bar{M} = \bar{M}$, a contradiction. \square

The following example shows that in general the converse of Theorem 2.13 is not true.

Example 2.14 Let K be a field, $D := K[\{x_i : i \in \mathbb{N}\}]$ (a unique factorization domain) and $R = K[\{x_i : i \in \mathbb{N}\}]/(\{x_i x_j : i, j \in \mathbb{N}\})$ where $(\{x_i x_j : i, j \in \mathbb{N}\})$ is the ideal of D

generated by the subset $\{x_i x_j : i, j \in \mathbb{N}\} \subseteq D$. Let $\bar{x}_k = x_k + (\{x_i x_j : i, j \in \mathbb{N}\})$ for each $k \in \mathbb{N}$ and $\mathcal{M} = (\{\bar{x}_k : k \in \mathbb{N}\})$. The ideal \mathcal{M} is simply the image of the maximal ideal $\mathcal{N} = (\{x_k : k \in \mathbb{N}\})$ of D . Clearly $\mathcal{M}^2 = (0)$ and so R is a local zero-dimensional ring, but is not Artinian (Noetherian). Since $\mathcal{M}^2 = (0)$, for each nonzero R -module M , $\mathcal{M}M \neq M$. Thus by Proposition 2.9 and Theorem 2.12, M is a \mathbb{P} -radical module. Thus every R -module is \mathbb{P} -radical, but R is not Artinian.

A ring R is called a Max-ring (or a Bass ring) if every nonzero R -module has a maximal submodule. Also, a ring R is called a P-ring if every nonzero R -module has a prime submodule. In [6], Theorem 3.9, it is shown that commutative P-rings coincide with Max-rings. Moreover, we have the following lemma which is essentially Theorem 2 of [13].

Lemma 2.15 *For a commutative ring R , the following conditions are equivalent:*

- (1) R is a max ring;
- (2) $R/J(R)$ is a regular ring and $J(R)$ is a t -nilpotent ideal.

The following theorem offer several characterizations of Noetherian rings R over which every module is \mathbb{P} -radical.

Theorem 2.16 *Consider the following statements for a ring R :*

- (1) R is an Artinian ring.
- (2) Every R -module is primeful.
- (3) Every R -module is \mathbb{P} -radical.
- (4) Every R -module is \mathbb{M} -radical.
- (5) R is a Max-ring.
- (6) R is a P-ring.
- (7) $\dim(R) = 0$.

Then $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Leftrightarrow (6) \Rightarrow (7)$. Consequently, when R is a Noetherian ring (or a domain), all the six statements are equivalent.

Proof. $(1) \Rightarrow (2)$ is by Theorem 2.13.

$(2) \Rightarrow (3)$ is by Proposition 2.3.

$(3) \Rightarrow (4)$ is clear.

$(4) \Rightarrow (5)$. Assume that every R -module is \mathbb{M} -radical. Let M be a nonzero R -module.

Then $\text{Ann}(M) \neq R$ and so there exists a maximal ideal \mathcal{M} of R such that $\text{Ann}(M) \subseteq \mathcal{M}$. Since M is maxful, there exists a prime submodule P of M with $(P : M) = \mathcal{M}$. Thus M/P is an R/\mathcal{M} -module (R/\mathcal{M} -vector space), and hence, M/P has a maximal R/\mathcal{M} -submodule, say K/P . It is easy to see that $K \not\subseteq M$ is a maximal R -submodule. Thus every nonzero R -module has a maximal submodule, i.e., R is a Max-ring.

(5) \Leftrightarrow (6) is by [6], Theorem 3.9.

(6) \Rightarrow (7). Let R be a P-ring. Suppose that \mathcal{P} is a non-maximal prime ideal of R . Set $R' = R/\mathcal{P}$ and let K be the field of fractions of R' . Since R' is not a field, $R' \neq K$ and K is a divisible R' -module. It follows that K/R' is a nonzero torsion divisible R' -module. Thus by Lemma 2.10, K/R' is a primeless R' -module. Now it is easy to see that K/R' is a primeless R -module. Thus R is not a P-ring, a contradiction.

(4) \Rightarrow (2). Assuming (4) to hold. We conclude from (4) \Rightarrow (7) that $\dim(R) = 0$. Theorem 2.12 now shows that every R -module is primeful.

Finally, if R is a Noetherian ring, then $\dim(R) = 0$ if and only if R is an Artinian ring. Thus (6) \Rightarrow (1), when R is a Noetherian ring. \square

The following is now immediate.

Corollary 2.17 *Let R be a domain. Then the following statements are equivalent:*

- (1) *Every R -module is primeful.*
- (2) *Every R -module is \mathbb{P} -radical.*
- (3) *Every R -module is \mathbb{M} -radical.*
- (4) *R is a field (i.e., R is an Artinian domain).*

The following proposition suggests that for a multiplication module M the four concepts “finitely generated”, “primeful”, “ \mathbb{P} -radical” and “ \mathbb{M} -radical” are equivalent.

Proposition 2.18 *Consider the following statements for a nonzero R -module M :*

- (1) *M is finitely generated.*
- (2) *M is primeful.*
- (3) *M is a \mathbb{P} -radical module.*
- (4) *$(\mathcal{P}M : M) = \mathcal{P}$ for every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$.*
- (5) *M is a \mathbb{M} -radical module.*

Then (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5). When M is a multiplication module, then (5) \Rightarrow (1).

Proof. (1) \Rightarrow (2) is by [18], Proposition 3.8.

(2) \Rightarrow (3) is by Proposition 2.3.

(3) \Rightarrow (4). Since $\mathcal{P} \subseteq (\mathcal{P}M : M) \subseteq (\sqrt[\mathcal{P}]{\mathcal{P}M} : M)$ for every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$, the proof is clear.

(4) \Rightarrow (5). For every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$ the equality $(\mathcal{P}M : M) = \mathcal{P}$ implies that $\mathcal{P}M \neq M$. Thus M is a \mathbb{M} -radical module.

When M is a multiplication module, then (5) \Rightarrow (1) is by [18], Proposition 3.8. \square

We conclude this section with the next analogue of Nakayama's Lemma.

Proposition 2.19 *Let M be a \mathbb{M} -radical R -module. Then M satisfies the following condition (NAK): If I is an ideal of R contained in the Jacobson radical $J(R)$ such that $IM = M$, then $M = (0)$.*

Proof. Suppose that $M \neq (0)$. Then $\text{Ann}(M) \neq R$. If \mathcal{M} is any maximal ideal containing $\text{Ann}(M)$, then $I \subseteq \mathcal{M}$ and $IM = M = \mathcal{M}M$, a contradiction. \square

3 Characterization of Semisimple \mathbb{P} -Radical Modules

Recall that for a module M , the *socle* of M (denoted by $\text{soc}(M)$) is the sum of all simple (minimal) submodules of M . If there are no minimal submodules in M we put $\text{soc}(M) = (0)$. Thus M is a semisimple module if $\text{soc}(M) = M$. Also, a semisimple module M is called homogeneous if any two simple submodules of M are isomorphic. It is easy to check that an R -module M is homogeneous semisimple if and only if $\text{Ann}(M)$ is a maximal ideal.

Next, we aim to characterize \mathbb{P} -radical semisimple modules. First we need the following definition.

Definition 3.1 . Let R be a ring. A semisimple R -module M is called *full semisimple* if for each maximal ideal $\mathcal{M} \supseteq \text{Ann}(M)$ the simple R -module R/\mathcal{M} may be embedded in M (i.e., there exists a submodule N of M such that $N \cong \bigoplus_{\text{Ann}(M) \subseteq \mathcal{M} \in \text{Max}(R)} R/\mathcal{M}$).

Example 3.2 Let $M = \bigoplus_{p \in \Omega} \mathbb{Z}/p\mathbb{Z}$ as a \mathbb{Z} -module where Ω is the set of prime integers. Clearly M is a full semisimple \mathbb{Z} -module. But the semisimple \mathbb{Z} -module $M_1 = \bigoplus_{2 \neq p \in \Omega} \mathbb{Z}/p\mathbb{Z}$ is not full semisimple since $\text{Ann}(M_1) = (0) \subseteq 2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ is not a submodule of M_1 .

The proof of the following result is straightforward and left to the reader.

Proposition 3.3 *Let M be a semisimple R -module such that $\text{Ann}(M)$ is a finite intersection of maximal ideals. Then M is full semisimple. In particular, all finitely generated semisimple modules as well as all homogenous semisimple modules are full semisimple.*

Lemma 3.4 *Let M be an R -module with nonzero socle. Then M is a prime module if and only if M is a homogeneous semisimple module (i.e., $\text{Ann}(M)$ is a maximal ideal).*

Proof. Let M be a prime module with nonzero socle and let Rm be a simple submodule of M . Then $\text{Ann}(m) = \text{Ann}(M) = P$ and hence P is a maximal ideal of R . Since $\text{Ann}(m) = \text{Ann}(m')$ for each $0 \neq m' \in M$, M is a homogeneous semisimple R -module. The converse is evident. \square

Now we are in position to show that the two concepts \mathbb{M} -radical and full semisimple are equivalent for semisimple modules.

Proposition 3.5 *Let M be a semisimple R -module. Then M is \mathbb{M} -radical if and only if M is full semisimple.*

Proof. Since M is a semisimple R -module, we can assume that $M = \bigoplus_{i \in I} R/\mathcal{M}_i$ where I is an index set and each \mathcal{M}_i is a maximal ideal of R .

(\Rightarrow). Clearly $\text{Ann}(M) = \bigcap_{i \in I} \mathcal{M}_i$. Suppose $\mathcal{M} \supseteq \text{Ann}(M)$ is a maximal ideal of R . If $\mathcal{M} \neq \mathcal{M}_i$ for each $i \in I$, then $\mathcal{M}(R/\mathcal{M}_i) = R/\mathcal{M}_i$ for each $i \in I$. It follows that $\mathcal{M}M = M$, contrary to (2) (see Proposition 2.9). Thus $\mathcal{M} = \mathcal{M}_i$ for some $i \in I$ and hence R/\mathcal{M} may be embedded in M .

(\Leftarrow). Suppose $\mathcal{M} \supseteq \text{Ann}(M)$ is a maximal ideal of R . Since M is full semisimple, $\mathcal{M} = \mathcal{M}_i$ for each $i \in I$ and so $\mathcal{M}M \neq M$. Thus by Proposition 2.9, M is a \mathbb{M} -radical module. \square

Proposition 3.6 *Let M be a semisimple R -module. Then the following statement are equivalent:*

- (1) M is a \mathbb{P} -radical module.
- (2) M is a \mathbb{M} -radical module and $R/\text{Ann}(M)$ is a Hilbert ring.
- (3) M is full semisimple and $R/\text{Ann}(M)$ is a Hilbert ring.

Proof. Since M is a semisimple R -module, we can assume that $M = \bigoplus_{i \in I} R/\mathcal{M}_i$ where I is an index set and each \mathcal{M}_i is a maximal ideal of R .

(1) \Rightarrow (2). Since every \mathbb{P} -radical module is \mathbb{M} -radical, it suffices to show that $R/\text{Ann}(M)$

is a Hilbert ring. Suppose $\mathcal{P} \supseteq \text{Ann}(M)$ is a prime ideal of R . Thus $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$ and hence $\sqrt[p]{\mathcal{P}M} \neq M$. We can assume that $\sqrt[p]{\mathcal{P}M} = \bigcap_{\lambda \in \Lambda} P_\lambda$, where Λ is an index set and each P_λ is a prime submodule of M containing $\mathcal{P}M$. Thus for each P_λ , the factor module M/P_λ is a prime semisimple module and hence by Lemma 3.4, M/P_λ is a homogenous semisimple R -module, i.e., $\mathcal{M}_\lambda := (P_\lambda : M)$ is a maximal ideal of R . Thus

$$\mathcal{P} = (\sqrt[p]{\mathcal{P}M} : M) = \left(\bigcap_{\lambda \in \Lambda} P_\lambda : M \right) = \bigcap_{\lambda \in \Lambda} (P_\lambda : M) = \bigcap_{\lambda \in \Lambda} \mathcal{M}_\lambda.$$

Thus every prime ideal $\mathcal{P} \supseteq \text{Ann}(M)$ is an intersection of maximal ideals of R , i.e., $R/\text{Ann}(M)$ is a Hilbert ring.

(2) \Leftrightarrow (3) is by Proposition 3.5.

(2) \Rightarrow (1) is by Theorem 2.11. \square

Corollary 3.7 *Let R be a Hilbert ring or a domain with $\dim(R) = 1$. For a semisimple R -module M the following statement are equivalent:*

- (1) M is a \mathbb{P} -radical module.
- (2) M is a \mathbb{M} -radical module.
- (3) M is full semisimple.

Proof. If R is a Hilbert ring then by Proposition 3.5 and Theorem 2.11, the proof is complete. Thus we can assume that R is a domain with $\dim(R) = 1$. Since M is a semisimple R -module, we can assume that $M = \bigoplus_{i \in I} R/\mathcal{M}_i$ where I is an index set and each \mathcal{M}_i is a maximal ideal of R .

(1) \Rightarrow (2) is clear.

(2) \Rightarrow (3) is by Proposition 3.5.

(3) \Rightarrow (1). Suppose $\mathcal{P} \supseteq \text{Ann}(M)$ is a prime ideal of R . Let $\mathcal{M} \supseteq \mathcal{P}$ be a maximal ideal. Since R is a domain with $\dim(R) = 1$, either $\mathcal{P} = (0)$ or $\mathcal{M} = \mathcal{P}$. If $\mathcal{M} = \mathcal{P}$, then $\mathcal{P} = \mathcal{M}_i$ for some i and so $\mathcal{P}M = \mathcal{M}_i M \neq M$. Clearly $\mathcal{P}M$ is a prime submodule of M with $(\mathcal{P}M : M) = \mathcal{P}$. It follows that $(\sqrt[p]{\mathcal{P}M} : M) = (\mathcal{P}M : M) = \mathcal{P}$. Now assume that $\mathcal{P} = (0)$. Then $\text{Ann}(M) = (0) = \bigcap_{i \in I} \mathcal{M}_i$. Since every proper submodule of a semisimple module is an intersection of maximal submodules and each maximal submodule is a prime submodule, we conclude that $\sqrt[p]{\mathcal{P}M} = \sqrt[p]{(0)} = (0)$. It follows that $(\sqrt[p]{(0)}M : M) = ((0) : M) = \text{Ann}(M) = (0) = \sqrt[p]{(0)}$. Thus M is a \mathbb{P} -radical module. \square

We conclude this paper with the following result that offer several characterizations for semisimple primeful modules.

Corollary 3.8 *Let M be a semisimple R -module. Then the following statements are equivalent:*

- (1) M is a primeful module.
- (2) M is a \mathbb{P} -radical module and $\dim(R/\text{Ann}(M)) = 0$.
- (3) M is a \mathbb{M} -radical module and $\dim(R/\text{Ann}(M)) = 0$.
- (4) M is a full semisimple module and $\dim(R/\text{Ann}(M)) = 0$.

Proof. (1) \Rightarrow (2). Since every primeful module is \mathbb{P} -radical, it suffices to show that $\dim(R/\text{Ann}(M)) = 0$. Suppose $\mathcal{P} \supseteq \text{Ann}(M)$ is a prime ideal of R . Thus there exists a prime submodule P of M such that $(P : M) = \mathcal{P}$. Since M/P is a prime semisimple R -module, by Lemma 3.4, \mathcal{P} is a maximal ideal. Thus $\dim(R/\text{Ann}(M)) = 0$.

(2) \Rightarrow (3) is by Theorem 2.12.

(3) \Rightarrow (4) is by Proposition 3.5. \square

Note: In Part II we shall continue the study of this construction.

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